STAT0041: Stochastic Calculus

Lecture 5 - Continuous-time Martingale

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Key concepts:

- *Right continuous martingale;*
- *Optional stopping theorem.*

5.1 Continuous-time martingale

We first introduce some conception in continuous-time stochastic process and assume filtered probability space satisfies usual conditionon in this lecture.

Definition 5.1 (Sample path) *Let $(X_t)_{t\in T}$ be a random process with values in E. The sample paths of X are the mappings $T \ni t \mapsto X_t(\omega)$ obtained when fixing $\omega \in \Omega$. The sample paths of X thus form a collection of mappings from T into E indexed by $\omega \in \Omega$.*

Definition 5.2 (Continuity of stochastic process) *We say a stochastic process is continuous, if almost surely all of its sample path is continuous. That is,*

$$\mathrm{P}\left(\{\omega | t \mapsto X_t(\omega) \text{ is continuous}\}\right) = 1.$$

Right/left-continuous can be defined similarly.

Sometimes it is not clear whether a stochastic process is continuous. We following show that, at the cost of "slightly" modifying the process, we can ensure that sample paths are continuous.

Definition 5.3 (Modification) *Let $(X_t)_{t\in T}$ and $(\tilde{X}_t)_{t\in T}$ be two random processes indexed by the same index set T and with values in the same metric space E. We say that $\tilde{X}$ is a **modification** of X if*

$$\forall t \in T, \quad \mathrm{P}(\tilde{X}_t = X_t) = 1.$$

Lemma 5.4 (Theorem 3.18 of [1]) *Let $(X_t)_{t\in T}$ be a supermartingale, such that the function $t \mapsto \mathbb{E}[X_t]$ is right-continuous. Then X has a modification with cadlag sample paths, which is also an $\mathscr{F}_t$-supermartingale. Especially, every martingale has a modification with cadlag sample paths because $\mathbb{E}[X_t]$ is a constant.*

Definition 5.5 (Stopping time) *A random variable $\tau : \Omega \to [0, \infty]$ is a stopping time of the filtration $\mathscr{F}_t$ if*

$$\{\tau \le t\} \in \mathscr{F}_t,$$

for every $t \ge 0$. The event field of the past before τ is then defined by

$$\mathscr{F}_\tau = \{A \in \mathscr{F}_\infty : \forall t \ge 0, A \cap \{\tau \le t\} \in \mathscr{F}_t\}.$$

Example 5.6 (First hitting time) *Let (X_t) be right-continuous $\mathscr{F}_t$ adapted process with state space E, and $A \subset E$, then* first hitting time to A

$$\tau_A(\omega) = \inf\{t > 0 : X_t(\omega) \in A\}$$

is a stopping time.

Example 5.7 *Let τ be a stopping time and let σ be an $\mathscr{F}_\tau$-measurable random variable with values in $[0, \infty]$, such that $\sigma \ge \tau$. Then σ is also a stopping time.*

In particular, if T is a stopping time,

$$T_n = \sum_{k=0}^{\infty} \frac{k+1}{2^n} \mathbf{1}_{\{k2^{-n} < T \le (k+1)2^{-n}\}} + \infty \cdot \mathbf{1}_{\{T = \infty\}}, \quad n = 0, 1, 2, \ldots$$

defines a sequence of stopping times that decreases to T.

Definition 5.8 (Martingale) *Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \ge 0}, \mathrm{P})$ be a filtered probability space, $X = (X_t)$ be a adapted process on $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathrm{P})$ satisfying $\mathbb{E}[|X_t|] < \infty$ is called*

*(1) a $\mathscr{F}_t$-**martingale** if $\forall\, 0 \le s < t$, $\mathbb{E}[X_t | \mathscr{F}_s] = X_s$;*

(2) a $\mathscr{F}_t$-supermartingale if $\forall\, 0 \le s < t$, $\mathbb{E}[X_t | \mathscr{F}_s] \le X_s$;

(3) a $\mathscr{F}_t$-submartingale if $\forall\, 0 \le s < t$, $\mathbb{E}[X_t | \mathscr{F}_s] \ge X_s$.

Proposition 5.9 (Doob's martingale inequality, Proposition 3.15 of [1])

Let (X_t) be a submartingale with right-continuous sample paths, then for all $c > 0$ and $T < \infty$,

$$c \cdot \mathrm{P}(\sup_{0 \le t \le T} X_t \ge c) \le \mathbb{E}[X_T^+].$$

Let (X_n) be a martingale with right-continuous sample paths, and for some $p \ge 1$, $\mathbb{E}|X_t|^p < \infty$. Then for all for all $c > 0$ and $T < \infty$,

$$\mathrm{P}(\sup_{0 \le t \le T} |X_t| \ge c) \le \frac{\mathbb{E}|X_t|^p}{c^p}.$$

5.2 Optional stopping theorem

Lemma 5.10 (Convergence theorem for supermartingales, Theorem 3.19 of [1])

Let X be a supermartingale with right-continuous sample paths. Assume that the collection $(X_t)_{t\geq 0}$ is bounded in L^1. Then there exists a random variable $X_\infty \in L^1$ such that

$$\lim_{t\to\infty} X_t = X_\infty, \quad a.s.$$

Lemma 5.11 (Doob's optional stopping theorem (uniformly integrable), Theorem 3.22 of [1])

Let (X_n) be a uniformly integrable martingale with right-continuous sample paths. Then, for every choice of the stopping times τ, σ such that $\tau \leq \sigma$, we have $X_\tau, X_\sigma \in L^1$ and

$$X_\tau = E[X_\sigma \mid \mathscr{F}_\tau]$$

Lemma 5.12 (Doob's optional stopping theorem (bounded case), Corollary 3.23 of [1])

Let $(X_t)_{t\geq 0}$ be a martingale with right-continuous sample paths and $\sigma \leq \tau < \infty$ be two almost surely bounded stopping times, then

$$\mathbb{E}[X_\tau | \mathscr{F}_\sigma] = X_\sigma.$$

Lemma 5.13 (Doob's sampling theorem, Corollary 3.24 of [1])

Let (X_t) be a martingale and τ be a stopping time both with respect to a filtration $(\mathscr{F}_t)$, then the stopped process X^τ is a martingale.

Moreover, if stopping time τ is bounded, then $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$.

References

[1] Le Gall, Jean-François. Brownian motion, martingales, and stochastic calculus. Springer International Publishing Switzerland, 2016.